Analysis of distribution of cosmic microwave background photon in terms of non-extensive statistics and formulas with temperature fluctuation

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Abstract

To take into account the temperature fluctuation in the Planck distribution, we calculate convolution integral with several probability distributions. Using these formula as well the Planck distribution and a formula in the non-extensive statistics, we analyze the data measured by the Cosmic Background Explorer (COBE). Our analysis reveals that the derivation from the Planck distribution is estimated as $|q-1| = 4.4 \times 10^{-5}$, where q means the magnitude of the non-extensivity or the temperature fluctuation, provided that the dimensionless chemical potential proposed by Zeldovich and Sunyaev exists. Comparisons of new formulas and the Planck distribution including the Sunyaev-Zeldovich (S-Z) effect are made.

I. INTRODUCTION

One of interesting subjects in thermodynamics is relating to the non-extensive statistics [1, 2, 3]. Several years ago, authors of Refs. [4, 5] showed that formulas in the non-extensive statistics, i.e., the temperature fluctuations, are calculated by the convolution integral with the gamma distribution [4] and a calculation with a distribution described by $\exp(-2|u|^\alpha)$, where α is fractional number [5]. In this report, we calculate the temperature fluctuation in the Planck distribution

$$D_{Planck}(\beta, \nu, \mu) = \frac{C_B \nu^3}{e^{\beta\omega + \mu} - 1} , \tag{1}$$

where $C_B = 8\pi h/c^3$, $\omega = h\nu$ and $\beta = 1/k_B T$. μ denotes the dimensionless chemical potential introduced by Zeldovich and Sunyaev [6]. Equation (1) is applied to the data measured by the Cosmic Background Explorer (COBE) [7, 8].

The formula in the non-extensive statistics is obtained in Ref. [2] as

$$F^{(NETD)}(\beta, \nu) = D_{Planck}(\beta, \nu)[1 - e^{-x}]^{(q-1)}$$
$$\times \left\{ 1 + (1-q)x \left[\frac{1 + e^{-x}}{1 - e^{-x}} - \frac{x}{2} \frac{1 + 3e^{-x}}{(1 - e^{-x})^2} \right] \right\} , \tag{2}$$

where $x = \beta\omega$. (It should be noticed that calculation in Ref. [2] is done without μ. The value of χ^2 is 108 and the degree of freedom is 31 [2]).

In the next section, we calculate the convolution integrals using the several probability distributions [9, 10, 11]. In the 3rd section we analyze the data reported in Ref. [12, 13] in terms of Eqs. (1) and (2) with μ, as well as new formulas. In the final section, concluding remarks are presented. We look for the origin of the temperature fluctuation.

II. CONVOLUTION INTEGRALS OF EQ. (1) WITH PROBABILITY DISTRIBUTIONS

First of all, according to Refs. [4, 5], we consider the gamma distribution for the temperature fluctuation as

$$P(\alpha, \beta, \beta_0) = \frac{1}{\Gamma(\alpha)} \left(\frac{\alpha}{\beta_0} \right)^\alpha \beta^{\alpha-1} e^{-\frac{\alpha}{\beta_0}\beta} \tag{3}$$

The formula, Eqs. (3), is also named the χ^2 distribution. The convolution integral is given as

$$F^{(gamma)}(\beta_0, \nu, \alpha) = \int_0^\infty d\beta P(\alpha, \beta, \beta_0) \times D_{Planck}(\beta, \nu, \mu) = \sum_{n=1}^\infty C_B \nu^3 e^{-n\mu} \frac{1}{(1 + \frac{\beta_0}{\alpha}\omega n)^\alpha}$$

$$\xrightarrow{\alpha \gg 1} \sum_{n=1}^\infty C_B \nu^3 e^{-n\mu - \beta_0 \omega n + \frac{1}{2}\beta_0^2 \omega^2 n^2 (q-1) + O((q-1)^2)} , \quad (4)$$

where $\alpha = \frac{1}{q-1}$ and $(q-1) \ll 1$ are assumed.

The following generalized χ^2 distribution is known as the non-central χ^2 distribution or the generalized Glauker-Lachs formula and Perina-McGill formula [9, 10, 11].

$$\left(\frac{\alpha}{p}\right) \left[\frac{\left(\frac{\beta}{\beta_0}\right)}{(1-p)}\right]^{\frac{\alpha-1}{2}} \exp\left[-\frac{\alpha(1-p)}{p} - \frac{\alpha}{p}\frac{\beta}{\beta_0}\right]$$

$$\times I_{(\alpha-1)}\left(2\sqrt{\frac{\beta}{\beta_0}\left(\frac{\alpha}{p}\right)^2 (1-p)}\right) , \quad (5)$$

where $p = \langle n_{th} \rangle / \langle n_{total} \rangle$ and I denotes the modified Bessel function. $\langle n_{th} \rangle$ does the thermal boson. Eq. (5) is the solution of the following Fokken-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \beta}\left[-\frac{1}{2}\alpha\eta(\beta - 1)\right]P + \frac{1}{2}\frac{\partial^2}{\partial \beta^2}\eta\beta P , \quad (6)$$

where α and η are parameters. This is named the stochastic process for birth and death process with the immigration. The convolution integral of Eqs. (1) and (5) is expressed as

$$F^{(NCC)}(\beta_0, \nu, \alpha, \mu, p)$$
$$= \sum_{n=0}^\infty C_B \nu^3 e^{-n\mu} \frac{1}{(1 + \frac{\beta_0}{\alpha}\omega n)^\alpha} e^{-\frac{\alpha(1-p)}{p} + \kappa} , \quad (7)$$

where

$$\kappa = \left[\beta_0 \left(\frac{\alpha}{p}\right)^2 (1-p)\right] \Big/ \left[\frac{\alpha}{p\beta_0} + \omega n\right].$$

As $p \to 1$ and $\alpha = \frac{1}{q-1} \gg 1$, Eq. (7) reduces to Eq. (4).

As the third case, we consider the Gaussian distribution.

$$P(V, \beta, \beta_0) = \frac{1}{\sqrt{2\pi V^2}} \exp\left[-\frac{(\beta - \beta_0)^2}{2V^2}\right] , \quad (8)$$

where V^2 denotes the variance. This distribution is relating to the solution of the Ornstein-Uhlenbeck stochastic process,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \beta}r\beta P + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial \beta^2} . \quad (9)$$
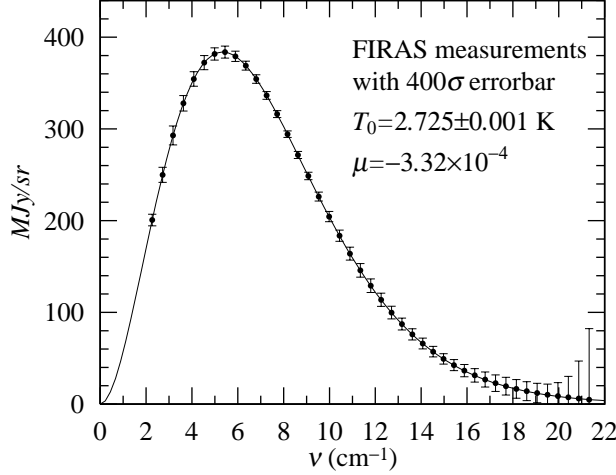
FIG. 1: Analysis of data by the COBE in terms of Eqs. (1). The data sets are obtained from T. C. Mather and D. J. Fixsen. See Refs [12, 13]. We use a method by D. Seibert [14] in calculation of error bar. The diagonal components are shown.

From the convolution integral of Eqs. (1) and (8), we obtain the following formula

$$F^{(Gauss)}(\beta_0, \nu, \alpha, \mu, V) = \sum_{n=1}^{\infty} C_B \nu^3 e^{-n\mu} e^{-n\beta_0\omega + \frac{n^2\omega^2}{2}V^2} . \tag{10}$$

When V^2 is expressed by $\beta_0^2(q-1)$, i.e., the temperature fluctuation, Eq. (10) reduces to Eq. (4)

III. ANALYSIS OF DATA BY COBE BY MEANS OF EQS. (1), (2), (4), (7) AND (10)

Using the CERN-MINUIT program, we can estimate values of parameters contained in formulas given in previous sections. In Fig. 1, we have confirmed the result reported in Refs. [12, 13]. Our results are shown in Table I. As seen in Table I, values of χ^2's for Eqs. (2) and (4) become to be somewhat better than that of the Planck distribution (Eq. (1)), because of the parameter q. The magnitude of $(q-1)$ denotes the degree of the deviation from the Planck distribution including μ, i.e. Eq. (1).

TABLE I: Analysis of data by the COBE by means of Eqs. (1), (2), (4), (7) and (10).

formula	$T_0 = 1/\beta_0$	$q - 1$	χ^2/n.d.f	μ	p
Eq. (1)	2.726 ± 0.001	—	141/41	0 (fixed)	—
(Planck)	2.725 ± 0.001	—	82/40	$(-3.32 \pm 0.43) \times 10^{-4}$	—
Eq. (2)	2.726 ± 0.001	$(2.00 \pm 0.07) \times 10^{-5}$	167/40	0 (fixed)	—
(NETD)	2.725 ± 0.001	$(4.67 \pm 1.60) \times 10^{-5}$	74/39	$(-5.16 \pm 0.77) \times 10^{-4}$	—
Eq. (4)	2.726 ± 0.001	$(2.00 \pm 0.06) \times 10^{-5}$	169/40	0 (fixed)	—
(gamma)	2.725 ± 0.001	$(4.40 \pm 1.50) \times 10^{-5}$	74/39	$(-5.16 \pm 0.76) \times 10^{-4}$	—
Eq. (7)	2.726 ± 0.001	$(-3.99 \pm 0.86) \times 10^{-5}$	119/39	0 (fixed)	0.9995 ± 0.6842
(NCC)	2.725 ± 0.001	$(4.40 \pm 1.62) \times 10^{-5}$	74/38	$(-5.16 \pm 0.74) \times 10^{-4}$	0.9995 ± 0.6641
Eq. (10)	2.726 ± 0.001	$(2.00 \pm 0.06) \times 10^{-5}$	169/40	0 (fixed)	—
(Gauss)	2.725 ± 0.001	$(4.40 \pm 1.47) \times 10^{-5}$	74/39	$(-5.16 \pm 0.74) \times 10^{-4}$	—

IV.   CONCLUDING REMARKS

We have calculated Eqs. (4), (7) and (10) and applied them to the data measured by the COBE [12, 13]. The following remarks are obtained from our analysis.

1) It is shown that the dimensionless chemical potential μ is playing an important role in Eq. (1), because of values of χ^2's.

2) The deviations from the Planck distribution including the chemical potential for the cosmic microwave background (CMB) photon is estimated by $|q - 1| = 4.4 \times 10^{-5}$.

   Both approaches, the non-extensive statistics and the convolution integrals, show the same magnitude.

3) As $(q-1) \ll 1$ and $V^2 = \beta_0(q-1)$ are assumed, it is difficult to distinguish the gamma distribution and the Gaussian distribution for the temperature fluctuation.

4) From $p \approx 1$ (in spite of large error bars) in Eq. (7) in Table I, it is known that the CMB photon is the pure thermal photon.

5) To look for the origin of the magnitude of the non-extensivity or the temperature fluctuation, we analyze the data by the following formula including the Sunyaev-Zeldovich

TABLE II: Analysis of data by the COBE in terms of Eq. (11).

formula	$T_0 = 1/\beta_0$	y	χ^2/n.d.f	μ
Eq. (11)	2.726 ± 0.001	$(-1.99 \pm 0.43) \times 10^{-5}$	$119/41$	0 (fixed)
	2.725 ± 0.001	$(2.20 \pm 0.75) \times 10^{-5}$	$74/40$	$(-5.16 \pm 0.76) \times 10^{-4}$

(S-Z) effect [6, 15],

$$F^{(S-Z\,effect)}(x, \beta, \mu, y) = D_{Planck}(\beta, \nu, \mu) + C_B \nu^3 \frac{xy e^x}{(e^x - 1)^2} \left[x \coth \frac{x}{2} - 4 \right], \quad (11)$$

where $x = \beta_0 \omega + \mu$. y is named the y parameter in the Compton scattering,

$$y = \int dl n_e \sigma_T \frac{k_B T_e}{m_e c^2}, \quad (12)$$

where l, n_e, σ_T and T_e are the size of region with high temperature in the cosmos, the number density of electrons, the cross section of Thomson scattering and the temperature of electron, respectively. Our estimated values are shown in Table II. The magnitude of y parameter $y \approx (2.2 \pm 0.75) \times 10^{-5}$ is almost the same as that of $|q - 1|$ in Table I. This coincidence suggests that Eqs. (4), (7) and (10) work well.

We summarize the following concerning the item 5): The deviations from the Planck distribution for the CMB photon are seen through the non-extensivity or the temperature fluctuation. Moreover, the physical origin is attributed to the Compton scattering between electrons in the galaxies and CMB photons [6, 15]. The behavior of the second term of Eq. (11) is shown in Fig. 2. The crossing point on the horizontal line(frequency) is determined by $x \coth \frac{x}{2} = 4$. Differences between Eqs. (2) and (4) and the Planck distribution with μ are also shown in Fig. 2. The small quantity, $(0.1\text{-}0.05)/400$, is reflecting the physical meaning of the inverse Compton scattering in the galaxies. It is observed through Eqs. (2) and (4).
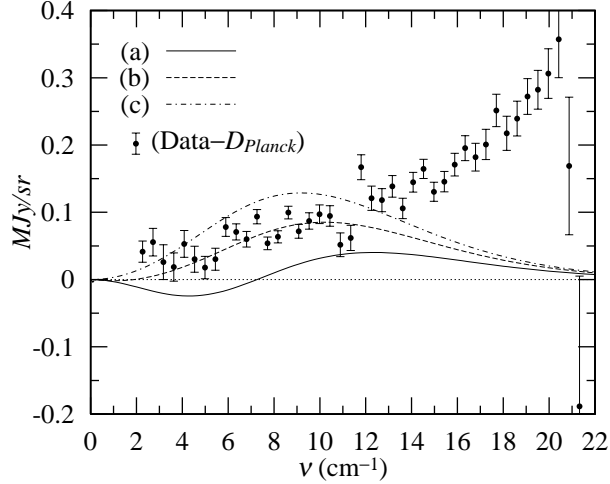
FIG. 2: Correction term to the Planck distribution. Planck distributions are subtracted at data points. (a) The second term of Eq. (11), i.e., Sunyaev-Zeldovich (S-Z) effect. (b) Difference between Eq. (2) and the Planck distribution with μ. (c) The same as (b) but Eq. (4) and the Planck distribution with μ.

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